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ON FINDING INTEGER SOLUTIONS TO THE POSITIVE PELL EQUATION

$y^2 = 40x^2 + 1$

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ABSTRACT:

This paper deals with the problem of obtaining non-zero distinct integer solutions to the no non-homogeneous binary quadratic equations with two unknowns $y^2 = 40x^2 + 1$. A few interesting properties among the solutions are given. The construction of second order Ramanujan Numbers is illustrated. Employing the linear combination among the solutions of the given equation, integer solutions for other choices of hyperbola & parabola and a few relations among special polygonal numbers are obtained.

KEYWORDS: Binary quadratic, Non-homogeneous quadratic, Pell equation, Positive Pell equation, Hyperbola. 2010 Mathematics Subject Classification: 11D09.

INTRODUCTION:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-13]. In this communication, yet another interesting hyperbola given by $y^2 = 40x^2 + 1$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$y^2 = 40x^2 + 1$ (1)

Whose smallest positive integer solutions by trial and error

$x_0 = 3 \,;\, y_0 = 19$

Whose general solution is given by

$x_n = \frac{1}{2\sqrt{40}} g_n \,;\, y_n = \frac{1}{2} f_n$

Where

$$f_n = \left(19 + 3\sqrt{40}\right)^{n+1} + \left(19 - 3\sqrt{40}\right)^{n+1}$$

$$g_n = (19 + 3\sqrt{40})^{n+1} - (19 - 3\sqrt{40})^{n+1}$$

Applying the Brahmagupta lemma between (x_0, y_0) & (x_n, y_n) the other solution of (2.1) are given by

$$x_{n+1} = \frac{19}{2\sqrt{40}} g_n + \frac{3}{2} f_n$$

$$y_{n+1} = \frac{19}{2} f_n + \frac{3\sqrt{40}}{2} g_n$$

The recurrence relations satisfied by *x* and *y* are given by

$$x_{n+2} - 38x_{n+1} + x_n = 0$$

$$y_{n+2} - 38y_{n+1} + y_n = 0$$

A few numerical examples are given in the following table.1

Table: 1 Numerical values.

n	x_n	y_n
0	3	19
1	114	721
2	4329	27379
3	164388	1039681
4	6242415	39480499
5	237047382	1499219281

From the above table we observe some interesting properties among the solutions which are presented below:

1. Relations between solutions

- $x_{n+3} - 38x_{n+2} + x_{n+1} = 0$
- $2y_{n+1} - \frac{1}{3}[2x_{n+2} - 38x_{n+1}] = 0$
- $2y_{n+2} - \frac{1}{3}[38x_{n+2} - 2x_{n+1}] = 0$
- $2y_{n+3} - \frac{1}{3}[1442x_{n+2} - 38x_{n+1}] = 0$
- $x_{n+2} - \frac{1}{38}[x_{n+1} + x_{n+3}] = 0$
- $y_{n+1} - \frac{1}{114}[x_{n+3} - 721x_{n+1}] = 0$
- $2y_{n+2} - \frac{1}{3}[x_{n+3} - x_{n+1}] = 0$
- $y_{n+3} - \frac{1}{114}[721x_{n+3} - x_{n+1}] = 0$
- $x_{n+2} - 19x_{n+1} - 3y_{n+1} = 0$
- $x_{n+3} - 721x_{n+1} - 114y_{n+1} = 0$
- $y_{n+2} - 19y_{n+1} - 120x_{n+1} = 0$
- $y_{n+3} - 721y_{n+1} - 4560x_{n+1} = 0$
- $x_{n+2} - \frac{1}{19}[x_{n+1} + 3y_{n+2}] = 0$
- $x_{n+3} - x_{n+1} - 6y_{n+2} = 0$
- $y_{n+1} - \frac{1}{19}[y_{n+2} - 120x_{n+1}] = 0$
- $y_{n+3} - \frac{1}{19}[721y_{n+2} + 120x_{n+1}] = 0$
- $x_{n+2} - \frac{1}{721}[19x_{n+1} + 3y_{n+3}] = 0$
- $x_{n+3} - \frac{1}{721}[x_{n+1} + 114y_{n+3}] = 0$
- $y_{n+1} - \frac{1}{721}[y_{n+3} - 4560x_{n+1}] = 0$
- $y_{n+2} - \frac{1}{721}[19y_{n+3} - 120x_{n+1}] = 0$

2. Each of the following expressions represents a Perfect Square

- $2y_{2n+1} + 2$
- $[2y_{2n+3} - 76y_{2n+2}] + 2$
- $\frac{1}{19}[2y_{2n+2} - 240x_{2n+1}] + 2$

- $[38y_{2n+2} - 240x_{2n+2}] + 2$
- $\frac{1}{19}[1442y_{2n+2} - 240x_{2n+3}] + 2$
- $\frac{1}{721}[2y_{2n+3} - 9120x_{2n+1}] + 2$
- $[2y_{2n+3} - 480x_{2n+2}] + 2$
- $[1442y_{2n+3} - 9120x_{2n+3}] + 2$
- $\frac{1}{3}[2x_{2n+2} - 38x_{2n+1}] + 2$
- $\frac{1}{114}[2x_{2n+3} - 1442x_{2n+1}] + 2$
- $\frac{1}{3}[38x_{2n+3} - 1442x_{2n+2}] + 2$

3. Each of the following expressions represents a Cubical Integers

- $2y_{3n+2} + 6y_n$
- $[2y_{3n+4} - 76y_{3n+3}] + 3[2y_{n+2} - 76y_{n+1}]$
- $\frac{1}{19}[[2y_{3n+3} - 240x_{3n+2}] + 3[(2y_{n+1} - 240x_n]]$
- $[38y_{3n+3} - 240x_{3n+3}] + 3[38y_{n+1} - 240x_{n+1}]$
- $\frac{1}{19}[[1442y_{3n+3} - 240x_{3n+4}] + 3[1442y_{n+1} - 240x_{n+2}]]$
- $\frac{1}{721}[[2y_{3n+4} - 9120x_{3n+2}] + 3[2y_{n+2} - 9120x_n]]$
- $[2y_{3n+4} - 480x_{3n+3}] + 3[2y_{n+2} - 480x_{n+1}]$
- $[1442y_{3n+4} - 9120x_{3n+4}] + 3[1442y_{n+2} - 9120x_{n+2}]$
- $\frac{1}{3}[[2x_{3n+3} - 38x_{3n+2}] + 3[2x_{n+1} - 38x_n]]$
- $\frac{1}{114}[[2x_{3n+4} - 1442x_{3n+2}] + 3[2x_{n+2} - 1442x_n]]$
- $\frac{1}{3}[[38x_{3n+4} - 1442x_{3n+3}] + 3[38x_{n+2} - 1442x_{n+1}]]$

4. Each of the following expressions represents a Bi-quadratic Integer

- $2y_{4n+3} + 8y_{2n+1} + 6$
- $[2y_{4n+5} - 76y_{4n+4}] + 4[2y_{2n+3} - 76y_{2n+2}] + 6$
- $\frac{1}{19}[[2y_{4n+4} - 240x_{4n+3}] + 4[2y_{2n+2} - 240x_{2n+1}]] + 6$
- $[38y_{4n+4} - 240x_{4n+4}] + 4[38y_{2n+2} - 240x_{2n+2}] + 6$

- $\frac{1}{19}[[1442y_{4n+4} - 240x_{4n+5}] + 4[1442y_{2n+2} - 240x_{2n+3}]] + 6$
- $\frac{1}{721}[[2y_{4n+5} - 9120x_{4n+3}] + 4[2y_{2n+3} - 9120x_{2n+1}]] + 6$
- $[2y_{4n+5} - 480x_{4n+4}] + 4[2y_{2n+3} - 480x_{2n+2}] + 6$
- $[1442y_{4n+5} - 9120x_{4n+5}] + 4[1442y_{2n+3} - 9120x_{2n+3}] + 6$
- $\frac{1}{3}[[2x_{4n+4} - 38x_{4n+3}] + 4[2x_{2n+2} - 38x_{2n+1}]] + 6$
- $\frac{1}{114}[[2x_{4n+5} - 1442x_{4n+3}] + 4[2x_{2n+3} - 1442x_{2n+1}]] + 6$
- $\frac{1}{3}[[38x_{4n+5} - 1442x_{4n+4}] + 4[38x_{2n+3} - 1442x_{2n+2}]] + 6$

5. Each of the following expressions represents a Quintic Integer

- $[2y_{5n+4} + 5(2y_n)^3 - 5(2y_n)]$
- $[2y_{5n+6} - 76y_{5n+5}] + 5[2y_{n+2} - 76y_{n+1}]^3 - 5[2y_{n+2} - 76y_{n+1}]$
- $\frac{1}{19}[2y_{5n+5} - 240x_{5n+4}] + 5[\frac{1}{19}(2y_{n+1} - 240x_n)]^3 - 5[\frac{1}{19}(2y_{n+1} - 240x_n)]$
- $[38y_{5n+5} - 240x_{5n+5}] + 5[38y_{n+1} - 240x_{n+1}]^3 - 5[38y_{n+1} - 240x_{n+1}]$
- $\frac{1}{19}[1442y_{5n+5} - 240x_{5n+6}] + 5[\frac{1}{19}(1442y_{n+1} - 240x_{n+2})]^3 - 5[\frac{1}{19}(1442y_{n+1} - 240x_{n+2})]$
- $\frac{1}{721}[2y_{5n+6} - 9120x_{5n+4}] + 5[\frac{1}{721}(2y_{n+2} - 9120x_n)]^3 - 5[\frac{1}{721}(2y_{n+2} - 9120x_n)]$
- $[2y_{5n+6} - 480x_{5n+5}] + 5[2y_{n+2} - 480x_{n+1}]^3 - 5[2y_{n+2} - 480x_{n+1}]$
- $[1442y_{5n+6} - 9120x_{5n+6}] + 5[1442y_{n+2} - 9120x_{n+2}]^3 - 5[1442y_{n+2} - 9120x_{n+2}]$
- $\frac{1}{3}[2x_{5n+5} - 38x_{5n+4}] + 5[\frac{1}{3}[(2x_{n+1} - 38x_n)]^3 - 5[\frac{1}{3}(2x_{n+1} - 38x_n)]$
- $\frac{1}{114}[2x_{5n+6} - 1442x_{5n+4}] + 5[\frac{1}{114}(2x_{n+2} - 1442x_n)]^3 - 5[\frac{1}{114}(2x_{n+2} - 1442x_n)]$
- $\frac{1}{3}[38x_{5n+6} - 1442x_{5n+5}] + 5[\frac{1}{3}[(38x_{n+2} - 1442x_{n+1})]^3 - 5[\frac{1}{3}(38x_{n+2} - 1442x_{n+1})]$

6. Construction of Second order Ramanujan numbers:

The Process of obtaining Second order Ramanujan numbers from suitable choices of x and y is illustrated through an example below

Consider,

$\mathbf{x_1 = 114}$

$= 114 \times 1 = 2 \times 57 = 3 \times 38 = 6 \times 19$

A = B = C = D

From A = B, Consider the relation

$$(114+1)^2+(2-57)^2=(114-1)^2+(2+57)^2$$

$$=> (115)^2+(-55)^2=(113)^2+(59)^2=\mathbf{16250}$$

From A = C, Consider the relation

$$(114+1)^2+(3-38)^2=(114-1)^2+(3+38)^2$$

$$=> (115)^2+(-35)^2=(113)^2+(41)^2=\mathbf{14450}$$

From A = D, Consider the relation

$$(114+1)^2+(6-19)^2=(114-1)^2+(6+19)^2$$

$$=> (115)^2+(-13)^2=(113)^2+(25)^2=\mathbf{13394}$$

From B = C, Consider the relation

$$(2+57)^2+(3-38)^2=(2-57)^2+(3+38)^2$$

$$=> (59)^2+(-35)^2=(-55)^2+(41)^2=\mathbf{4706}$$

From B = D, Consider the relation

$$(2+57)^2+(6-19)^2=(2-57)^2+(6+19)^2$$

$$=> (59)^2+(-13)^2=(-55)^2+(25)^2=\mathbf{3650}$$

From C = D, Consider the relation

$$(3+38)^2+(6-19)^2=(3-38)^2+(6+19)^2$$

$$=> (41)^2+(-13)^2=(-35)^2+(25)^2=\mathbf{1850}$$

Each of the numbers 16250, 14550, 13394, 4706, 3650, 1850 is expressed as Sum of two squares in two different ways. Thus, the above numbers are Second order Ramanujan numbers.

REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in table 2 below

Table: 2 Hyperbola.

S.No	Hyperbola	(X,Y)
1.	$360X^2-Y^2=360$	$X=y_n$

		$Y = (y_{n+1} - 19y_n)$
2.	$519840X^2 - Y^2 = 519840$	$X = y_n$ $Y = (y_{n+2} - 721y_n)$
3.	$X^2 - 40Y^2 = 1$	$X = y_n$ $Y = x_n$
4.	$361X^2 - 40Y^2 = 361$	$X = y_n$ $Y = (x_{n+1} - 3y_n)$
5.	$519841X^2 - 40Y^2 = 519841$	$X = y_n$ $Y = (x_{n+2} - 114y_n)$
6.	$360X^2 - Y^2 = 360$	$X = (y_{n+2} - 38y_{n+1})$ $Y = (19y_{n+2} - 721y_{n+1})$
7.	$X^2 - 14440Y^2 = 361$	$X = (y_{n+1} - 120x_n)$ $Y = x_n$
8.	$X^2 - 40Y^2 = 1$	$X = (19y_{n+1} - 120x_{n+1})$ $Y = (19x_{n+1} - 3y_{n+1})$
9.	$X^2 - Y^2 = 361$	$X = (721y_{n+1} - 120x_{n+2})$ $Y = (19x_{n+2} - 114y_{n+1})$
10.	$X^2 - 20793640Y^2 = 519841$	$X = (y_{n+2} - 4560x_n)$ $Y = x_n$
11.	$361X^2 - 40Y^2 = 361$	$X = (y_{n+2} - 240x_{n+1})$ $Y = (721x_{n+1} - 3y_{n+2})$
12.	$X^2 - 40Y^2 = 1$	$X = (721y_{n+2} - 4560x_{n+2})$ $Y = (721x_{n+2} - 114y_{n+2})$
13.	$X^2 - 360Y^2 = 9$	$X = (x_{n+1} - 19x_n)$ $Y = x_n$
14.	$X^2 - 519840Y^2 = 12996$	$X = (x_{n+2} - 721x_n)$ $Y = x_n$
15.	$X^2 - 360Y^2 = 9$	$X = (19x_{n+2} - 721x_{n+1})$ $Y = (38x_{n+1} - x_{n+2})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table 3 below

Table: 3 Parabola

S. No	Parabola	(X,Z)
1.	$360X - 2Z^2 = 360$	$X = y_{2n+1}$ $Z = (y_{n+1} - 19y_n)$
2.	$519840X - 2Z^2 = 519840$	$X = y_{2n+1}$ $Z = (y_{n+2} - 721y_n)$
3.	$X - 80Z^2 = 1$	$X = y_{2n+1}$ $Z = x_n$
4.	$361X - 80Z^2 = 361$	$X = y_{2n+1}$ $Z = (x_{n+1} - 3y_n)$
5.	$519841X - 80Z^2 = 519841$	$X = y_{2n+1}$ $Z = (x_{n+2} - 114y_n)$
6.	$360X - 2Z^2 = 360$	$X = (y_{2n+3} - 38x_{2n+2})$ $Z = (38y_{n+2} - 72y_{n+1})$
7.	$X - 3040Z^2 = 19$	$X = (y_{2n+2} - 120x_{2n+1})$ $Z = x_n$
8.	$X - 80Z^2 = 1$	$X = (19y_{2n+2} - 120x_{2n+2})$ $Z = (19x_{n+1} - 3y_{n+1})$
9.	$19X - 80Z^2 = 722$	$X = (721y_{2n+2} - 120x_{2n+3})$ $Z = (19x_{n+2} - 114y_{n+1})$
10.	$X - 80Z^2 = 722$	$X = (y_{2n+3} - 4560x_{2n+1})$ $Z = x_n$
11.	$361X - 80Z^2 = 361$	$X = (y_{2n+3} - 240x_{2n+2})$ $Z = (721x_{n+2} - 3y_{n+2})$
12.	$X - 80Z^2 = 1$	$X = (721y_{2n+3} - 4560x_{2n+3})$ $Z = (721x_{n+2} - 114y_{n+2})$
13.	$X - 240Z^2 = 3$	$X = (x_{2n+2} - 19x_{2n+1})$ $Z = x_n$
14.	$X - 9120Z^2 = 114$	$X = (x_{2n+3} - 721x_{2n+1})$ $Z = x_n$
15.	$X - 480Z^2 = 9$	$X = (19x_{2n+3} - 721x_{2n+2})$ $Z = (38x_{n+1} - 2x_{n+2})$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 40x^2 + 1$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive pell equations and determine the integer solutions along with suitable properties.

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