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ANALYSIS OF THE WAVE FUNCTION IN THE EQUATION OF SCHRODINGER, ANOTHER AND EXAMPLE.IN SEARCH OF THE MEANING OFWAVE FUNCTION

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SUMMARY

This paper presents a theoretical review.Presenting references such as the Basel problem, addressing how to apply the Schrödinger equation to a wave function. The intention is to reference the geometric construction that Euler used to build his solution to the Basel problem. Likewise, a brief reference to Trigonometric Transformations, such as Integration by Parts in Differential Calculus and Integral Calculus, is also included. With this, one has the condition... In order to create epistemological and didactic conditions for solving exercises with the Wave Function in the Schrödinger Equation, it is known in University Education that the Wave Function is calculated in the Schrödinger Equation; however, the necessary epistemological analysis is not given, such as the Analysis of the Wave Function, that is, What is it? Or What is the Meaning of this Wave Function? With a Mathematical and Conceptually Epistemological approach in Physics. This article is a starting point as a reference for conceptual approach and analysis with various mathematical tools.

KEYWORDS: Wave Function, Euler, ProblemFrom Basileira, Trigonometric Transformations, Pythagorean School, Platonic Geometry.

INTRODUCTION

In light of the points raised in previous articles, such asTHEanalysis of minimum uncertainty states through series mathematics, conjectures and mathematical sequences according to the matrices of heisenberg, born and jordan of quantum mechanics[2], A study from the operator of evolution and the descriptions ofHeisenberg, ofSChrodinger and OfInteraction AndIts verifications inSseries through powerSeriesANDxpansions, concomitantly with the integration of h(t) in andANDexponential, for xample,Function theRiemannZeta[3].Applying Lucas Succession to the Harmonic Hamiltonian of the Representation of Born and Jordan[5].It was in 1735 that Euler gained international fame after solving a famous problem in quantum mechanics.uThe method, first proposed by Pietro Mengoli, of finding the exact value of the sum[4]

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots$$

This article aims to establishandto analyze wave functions, emphasizing that the theoryfrom theSchrödinger's equation (and its analogy to Newton's Second Law) and the wave function should be covered in university-level quantum mechanics textbooks, such as...GRIFFITHS, David J.Introduction to quantum mechanics. 3rd ed. Cambridge: Cambridge University Press, 2018[1].Reference is made in the presentI work with the wave function, which is nothing more than...thatFinally, a particle, by nature, is located at a point, while the wave function (as its name suggests) is distributed in space (it is a function of x, for any instant).Given t). How does such an object represent the state of a particle? The answer is provided by Born's statistical interpretation of the wave function, in

$$|\psi(x,t)|^{2}$$

For example, suppose that we wish to normalize the wavefunction of a Gaussian wave-packet, centered on x=x0 , and of characteristic width sigma (see Section [s2.9]): that is,

$$\psi(x) = \psi_0 \, e^{-(x-x_0)^2/(4\sigma^2)}.$$

In order to determine the normalization constant , we simply substitute: ψ_0,
Equation ([e3.5]) into Equation ([e3.4]) to obtain

$$|\psi_0|^{2}\int_{-\infty}^{\infty} e^{-(x-x_0)^2/(2\sigma^2)}\,dx = 1.$$

Changing the variable of integration to

$$y = (x-x_0)/(\sqrt{2}\,\sigma)$$

, we get

$$|\psi_0|^2\sqrt{2}\,\sigma\int_{-\infty}^{\infty} e^{-y^2}\,dy = 1.$$

However,

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi},$$

which implies that

$$|\psi_0|^2 = \frac{1}{(2\pi\,\sigma^2)^{1/2}}.$$

Hence, a general normalized Gaussian wavefunction takes the form

$$\psi(x) = \frac{e^{i\,\varphi}}{(2\pi\ \sigma^2)^{1/4}}\,e^{-(x-x_0)^2/(4\,\sigma^2)},$$

where

$$\varphi \text{ i}$$

is an arbitrary real phase-angle.

It is important to demonstrate that if a wavefunction is initially normalized then it stays normalized as it evolves in time according to Schrödinger's equation. If this is not the case then the probability interpretation of the wavefunction is untenable, because it does not make sense for the probability that a measurement of yields any possible outcome (which is, manifestly, unity) to change in time. Hence, we require that

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\psi(x,t)|^2\,dx = 0,$$

This work uses another common reference, applicable to any wave function situation involving the Schrödinger equation, namely the Basel Problem, which can be found in thousands of books in online libraries worldwide.whatIt was in 1735 that Euler gained international fame after solving a famous problem in quantum mechanics.uThe method, first proposed by Pietro Mengoli, of finding the exact value of the sum[4]

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

of property 2.2[4]Euler applied the formula to the infinite roots of an infinite polynomial, obtaining...

$$-\frac{-\frac{1}{3!}}{1} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots + \frac{1}{n^2\pi^2} + \cdots$$

whereupon multiplying both sides by π^2 leads to

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$$

which was the solution presented by Euler for the Basel Problem.[4],[6] and [7].

These references show, in some way...The geometric construction of the Basel problem and the solution presented by Euler form an assumption and starting point for constructing the manipulations that should be performed.to be done at FuncatheWave equation for the Schrodinger equation. The main goal would be the analysis of the Wave Function with theBell's discovery effectively eliminated agnosticism as a viable option and transformed the decision of whether 1 or 2 is the correct choice into an experimental one, when you will have better conditions to appreciate the argument.uBell's theory; for now, it is sufficient to say that the experiments have decisively confirmed the orthodox interpretation: a particleIt simply doesn't have a precise position before the measurement, cthebehaving almost like waves on a lake; it is the measurement process thatinsistsin a given number and, thus, in a certain way, creates the specific result, limited only by the statistical weighting imposed by the waveform.[1].

The geometric construction of the Pythagorean school and the geometry of the School of Athens of the philosopher Plato serve as examples of starting points of reference.In conclusion, how

For example, suppose that we wish to normalize the wavefunction of a Gaussian wave-packet, centered on $x = x_0$, and of characteristic width σ (see Section [s2.9]): that is,

$$\psi(x) = \psi_0 \, e^{-(x-x_0)^2/(4\sigma^2)}. \tag{3.2.3}$$

In order to determine the normalization constant ψ_0, we simply substitute Equation ([e3.5]) into Equation ([e3.4]) to obtain

$$|\psi_0|^2 \int_{-\infty}^{\infty} e^{-(x-x_0)^2/(2\sigma^2)}\, dx = 1. \tag{3.2.4}$$

Changing the variable of integration to $y = (x - x_0)/(\sqrt{2}\,\sigma)$, we get

$$|\psi_0|^2 \sqrt{2}\,\sigma \int_{-\infty}^{\infty} e^{-y^2}\, dy = 1. \tag{3.2.5}$$

However ,

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}, \tag{3.2.6}$$

which implies that

$$|\psi_0|^2 = \frac{1}{(2\pi\sigma^2)^{1/2}}. \tag{3.2.7}$$

Hence, a general normalized Gaussian wavefunction takes the form

$$\psi(x) = \frac{e^{i\,\varphi}}{(2\pi\,\sigma^2)^{1/4}}\, e^{-(x-x_0)^2/(4\sigma^2)}, \tag{3.2.8}$$

where φ is an arbitrary real phase-angle.

Theoretical Framework:

The Statistical Interpretation of the Wave Function:

But what exactly is this 'wave function' and what does it do for you when obtained? After all, a particle, by nature, is located at a point, while the wave function (as its name suggests) is distributed in space (it is a function of x, for any instant).Given t). How does such an object represent the state of a particle? The answer is provided by Born's statistical interpretation of the wave function, in

$$|\psi(x,t)|^2$$

eh is the probability of finding the particle at point x, at time t, or, more precisely,

$$\int_a^b |\Psi(x,t)|^2 \, dx = \left\{ \begin{array}{l} \text{probabilidade de encontrar a partícula} \\ \text{entre } a \text{ e } b \text{ no instante } t. \end{array} \right\}$$

Probability is the area under the graph of

$$|\psi(x,t)|^2$$

For the wave function in Figure 1.2, it is quite likely that you will find the particle in the vicinity of point A, where

$$|\psi(x,t)|^2$$

It's large, and it's relatively unlikely you'll find it near point B.

Statistical interpretation presents aThis type of indeterminacy exists within quantum mechanics because, even if you know everything the theory has to say about a particle (that is, its wave function), you cannot accurately predict the outcome of a simple experiment to measure its position. All that quantum mechanics has to offer is statistical information about possible outcomes. This indeterminacy has deeply troubled both physicists and philosophers, and it is natural to question whether it is a fact of nature or a flaw in the theory.

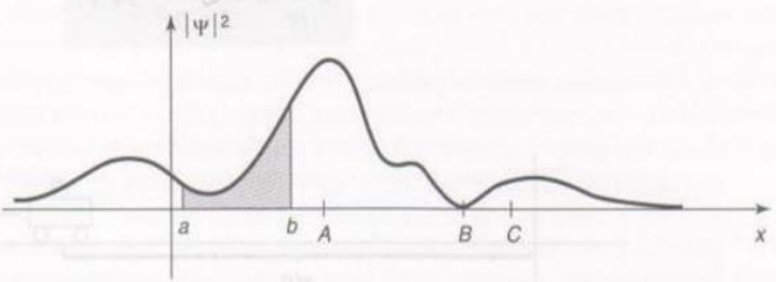

FIGURA 1.2 Função de onda típica. A área sombreada representa a probabilidade de encontrarmos a partícula entre a e b. A partícula provavelmente seria encontrada próximo a A, e, dificilmente, próximo a B.

Suppose I actually measure the particle's position and find that it is at point C. Question: where was the particle before I made the measurement? There are three plausible answers to this question, and they serve to distinguish the main schools of thought related to quantum indeterminacy:

1- The realist position: the particle was at C. This seems to be a sensible answer, and was defended by Einstein. Note, however, that if this is true, then quantum mechanics could not tell us this. For the realist, indeterminacy is not a fact of nature.But this is a reflection of our ignorance. As d'Espaganat stated, 'the position of the particle was never indeterminate, but merely unknown to the experimenter'. Evidently, \Psi does not encompass everything; some extra information (known as hidden variables) is necessary to have a complete description of the particle.

2-The Orthodox Position: the particle was nowhere. It was the act of measuring that forced the particle to 'make a decision' (but we will not dare to ask how and why it decided on point C). Jordan stated further that Bell's discovery effectively eliminated agnosticism as a viable option and transformed the decision of whether 1 or 2 is the correct choice into an experimental question. I will return to this story at the end of this book, when you will be in a better position to appreciate Bell's argument; for now, it is sufficient to say that the experiments decisively confirmed the orthodox interpretation: a particleIt simply doesn't have a precise position before measurement, behaving almost like waves on a lake; it's the measurement process that...insistsin a certain number and, thus, in a certain way, creates the specific result, limited only by the statistical weighting imposed by the waveform. In short: 'Observations not only disturb what is to be measured, but produce what is to be measured… We force (the particle) to assume a definite position'. This view (called the Copenhagen interpretation) is associated with Bohr and his followers. Among physicists, this has always been the most widely accepted position. Note, however, that if it is correct, there is something very peculiar about the act of measuring – something that more than half a century of discussions has clarified little.

3- The agnostic position: refuses to answer. This is not as foolish as it seems; after all, what sense can there be in making statements about the state of a particle before a measurement, when the only way to know if you are right is precisely by performing a measurement and, in that case, the result has nothing to do with 'before the measurement'? It is metaphysics (in the pejorative sense of the word) to worry about it.with something that cannot, by its nature, be tested. Pauli said: 'We should not torture ourselves trying to solve a problem about something we don't know exists or not, just as it is useless to try to solve the ancient problem of how many angels can sit on the head of a needle.' For decades, this was the 'defensive' position adopted by most physicists: they tried to sell the orthodox answer, but if you were persistent, they would retreat to the agnostic answer.They ended the conversation.

Until very recently, all three positions (realist, orthodox, and agnostic) had supporters.But in 1964, John Bell surprised the physics community by showing that there is an observable difference whether the particle had a precise (though unknown) position prior to measurement or not. Bell's discovery effectively eliminated agnosticism as a viable option and transformed the decision of whether 1 or 2 is the correct choice into an experimental question. I will return to this story at the end of this book, when you will be better able to appreciate Bell's argument; for now, it is sufficient to say that experiments have decisively confirmed the orthodox interpretation: a particle...It simply doesn't have a precise position before measurement, behaving almost like waves on a lake; it's the measurement process that...insistsin a given number and, thus, in a certain way, creates the specific result, limited only by the statistical weighting imposed by the waveform.

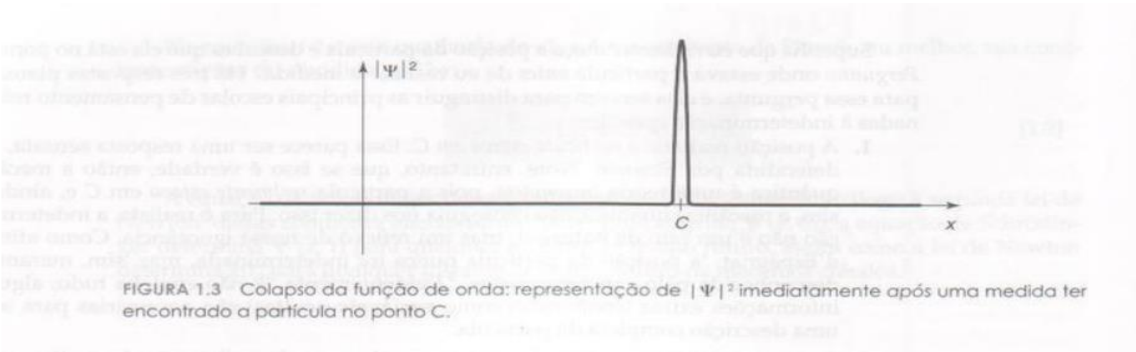

FIGURA 1.3 Colapso da função de onda: representação de |Ψ|² imediatamente após uma medida ter encontrado a partícula no ponto C.

What if IIf a second measurement were taken immediately after the first, would the result be C again, or would the act of measuring give us a completely new number each time? On this question, everyone agrees: a repetitive measurement (of the same particle)It should always result in the same value. In fact, it would be difficult to prove that the particle was...reallyfound at C at the first moment if this cannot be confirmed by an immediate repetition of the measurement, how does the orthodox interpretation explain the fact that the second measurement is conditional on repeating the value C? Evidently, the first measurement radically alters the wave function, so that it is now a narrow peak centered atC (Figure 1.3). We say that the wave function collapses, due to the measurement, into a peak at point C (and then disperses again, according to the Schrödinger equation, and therefore,(the second measurement must be done quickly). There are, then, two completely distinct types of physical processes: the 'common' ones, in which the wave functions evolve slowly governed by the Schrodinger equation, and the 'measured' ones, in which \Psi collapses suddenly and discontinuously.

The Basel Problem[4][6][7]:

It was in 1735 that Euler gained international fame after solving a famous problem in quantum mechanics.uThe method, first proposed by Pietro Mengoli, of finding the exact value of the sum[4]

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

showing that the series converges to the value π^2\6, a fact that surprised everyone, since the answer contained relationwOne of the most important constants in the history of mathematics. Many mathematicians of the time worked on this problem, including Jacques Bernoulli, but without success. Finally, the problem became known as the Basel Problem, in honor of Euler's hometown.[4].

This problem andhAttributed to the Italian mathematician Pietro Mengoli, who lived in the mid-18th century and proposed it in 1644. Several mathematicians attempted to solve it, but without success.

Namely, Jakob Bernoulli (1654-1705), Johann Bernoulli (1667-1748), John Wallis (1616-1703), Gottfried Wilhelm Leibniz (1646-1716), among others. According to Gayo and Wilhem (2015)[5], Jakob Bernoulli pondered the problem for some time, but without success. Upon his death, his brother Johann Bernoulli took over his professorship at the University of Basel, where he had the privilege of having Euler as a student.[4].

Certainly, in one of those classes he presented the problem to him. Some time later, Leonhard Euler arrived at the solution, which made him known and respected by the mathematical community. The problem consists of finding the sum of the inverses of the squares of the natural numbers.The sum of the reciprocals of natural numbers behaves as follows.

$$\begin{aligned}\sum_{n\in\mathbb{N}}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\cdots\\ &= 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16})+\cdots\\ &> 1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16})+\cdots\\ &= 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.\end{aligned}$$

It is not difficult to conclude that the sum of infinitely many terms equal to 1/2 is infinite, and since the sum of the reciprocals of squares is even greater than this, it is also infinite.

One way to see that the sum in the Basel Problem is finite is through the argument that the square of a natural number is greater than its product with its predecessor and less than its product with its successor; that is, if n is a natural number, then...

$$(n-1)n < n^2 < n(n+1).$$

It follows from this inequality that

$$\frac{1}{(n+1)n} < \frac{1}{n^2} < \frac{1}{n(n-1)}$$

which can be rewritten as

$$\frac{1}{n}-\frac{1}{n+1} < \frac{1}{n^2} < \frac{1}{n-1}-\frac{1}{n}.$$

Analysis of Minimum Uncertainty States through Series Mathematics, Conjectures and Mathematical Sequences according to the Matrices of Heisenberg, Born and Jordan of Quantum Mechanics..

Since this last inequality holds for every natural number n greater than 1, we have

$$\frac{1}{2} - \frac{1}{3} < \frac{1}{2^2} < 1 - \frac{1}{2},$$
$$\frac{1}{3} - \frac{1}{4} < \frac{1}{3^2} < \frac{1}{2} - \frac{1}{3},$$
$$\frac{1}{4} - \frac{1}{5} < \frac{1}{4^2} < \frac{1}{3} - \frac{1}{4},$$
$$\vdots$$
$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}.$$

Adding these inequalities together, member by member, we obtain

$$\frac{1}{2} - \frac{1}{n+1} < \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 - \frac{1}{n}.$$

Adding 1 to all members gives us

$$\frac{3}{2} - \frac{1}{n+1} < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

At this point, the reader familiar with arguments from differential and integral calculus should conclude that the larger the value of n, the closer to zero the terms 1/n + 1 and 1/n become, leading us to conclude that the value of the sum in the Basel Problem lies between 1.5 and 2. With the aid of a calculator, we can arrive at an approximate value of 1.645.

Euler's Proof

To understand the idea used by Euler to find the value of the sum in the Basel Problem, it is first necessary to understand some mathematical arguments and concepts, such as the expansion of functions in Taylor and/or Maclaurin series, the Fundamental Trigonometric Limit, among others.

The first mathematical concept, addressed in this test, ishthat a function f can be written as an infinite polynomial, that is,

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$$

When the function f is infinitely differentiable in a neighborhood of the point x0 with the coefficients ak depending on the derivatives of f at x0. For more details, it is suggested to read about the Taylor Series expansion, which can be found in books on Differential and Integral Calculus. As a particular case of the Taylor Series expansion, there is the Maclaurin Series expansion, where the expansion around x0 = 0 is considered, and in this case, it is written...

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots .$$

As an example, consider the Maclaurin series expansion for the function f(x) = sin x used by Euler in his proof. Since fEhAn infinitely differentiable form is obtained by differentiating vtheseveral times,

$$\begin{aligned}
f(x) &= \text{sen } x = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots , \\
f'(x) &= \cos x = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots , \\
f''(x) &= -\text{sen } x = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots + n(n-1)x^{n-2} + \cdots , \\
f'''(x) &= -\cos x = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4x + \cdots + n(n-1)(n-2)x^{n-3} + \cdots , \\
&\vdots
\end{aligned}$$

and, evaluating each equality at x = 0, the coefficients are found.

$$\begin{aligned}
a_0 &= \text{sen } 0 = 0 \\
a_1 &= \cos 0 = 1 \\
2a_2 &= -\text{sen } 0 = 0 \\
3 \cdot 2a_3 &= -\cos 0 = -1 \\
4 \cdot 3 \cdot 2a_4 &= -\cos 0 = 0 \\
5 \cdot 4 \cdot 3 \cdot 2a_5 &= -\cos 0 = 1 \\
6 \cdot 5 \cdot 4 \cdot 3 \cdot 2a_6 &= -\cos 0 = 0 \\
7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2a_7 &= -\cos 0 = -1 \\
&\vdots
\end{aligned}$$

Thus, we obtain

$$a_{2k} = 0 \quad \text{e} \quad a_{2k-1} = \frac{(-1)^{k+1}}{(2k-1)!}$$

as coefficients of the expansion of the sine function around x = 0 and is written

$$\text{sen}\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Substituting x with √x in the last equation and dividing both sides by √x gives us

$$\frac{\text{sen}\ \sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots .$$

Note that the function from property 2.2 for the infinite roots of an infinite polynomial, Euler applied it, obtaining...

$$g(x) = \frac{\text{sen}\ \sqrt{x}}{\sqrt{x}}$$

cancels out in

$$\pi^2, 4\pi^2, 9\pi^2, \cdots, n^2\pi^2, \cdots,$$

That is, g(x) has an infinite number of roots.

On the other hand, regarding a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$$

of degree $n \geq 1$ it was already known that these could be factored as

$$p(x) = (x - r_1)(x - r_2)(x - r_3) \cdots (x - r_{n-1})(x - r_n)$$

where r1, r2, r3, · · · , rn are all the roots of p(x). Note that

$$a_0 = p(0) = (-1)^n r_1 r_2 \cdots r_{n-1} r_n$$

$$\begin{aligned} a_1 = p'(0) = (-1)^{n-1} & r_2 r_3 \cdots r_{n-1} r_n + \\ & +(-1)^{n-1} r_1 r_3 \cdots r_{n-1} r_n + \\ & +(-1)^{n-1} r_1 r_2 r_4 \cdots r_{n-1} r_n + \\ & + \cdots + (-1)^{n-1} r_1 r_2 \cdots r_{n-2} r_n \\ & +(-1)^{n-1} r_1 r_2 \cdots r_{n-2} r_{n-1} \end{aligned}$$

from which we obtain the following property relating the coefficients and roots of a polynomial of degree n,

$$-\frac{a_1}{a_0} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_{n-1}} + \frac{1}{r_n}.$$

Returning to the analysis of the function

$$g(x) = \frac{\text{sen } \sqrt{x}}{\sqrt{x}}$$

From property 2.2 for the infinitely many roots of an infinite polynomial, Euler applied it, obtaining...

$$-\frac{-\frac{1}{3!}}{1} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots + \frac{1}{n^2\pi^2} + \cdots$$

whereupon multiplying both sides by π^2 leads to

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$$

which was the solution presented by Euler for the Basel Problem. This solution ishThe passage from 1735 where he used property 2.2 remained unjustified for infinitely many roots for about 150 years. During this period, many mathematicians celebrated the brilliant idea developed by Euler in his proof but, at the same time, drew attention to the lack of rigor in this passage. Other mathematicians...theScientists obtained other solutions to the Basel Problem once the value for which the sum converged was known. In 1855, with the proof of the Factor TheoremwIn contrast to Weierstrass's argument, the passage can be justified by celebrating the validity of Euler's proof. Namely, Weierstrass's Factorization Theorem states that every integer function can be represented as a (possibly infinite) product involving its zeros. This theorem can be seen as an extension of the Factorization Theorem to polynomials of finite degree.

However, this is a necessary condition for the integral on the left-hand side of Equation ([e3.4]) to converge. Hence, we conclude that all wavefunctions that are *square-integrable* [i.e., are such that the integral in Equation ([e3.4]) converges] have the property that if the normalization condition ([e3.4]) is satisfied at one instant in time then it is satisfied at all subsequent times.

It is also possible to demonstrate, via very similar analysis to that just described, that

$$\frac{dP_{x \in a:b}}{dt} + j(b,t) - j(a,t) = 0,$$

where P is defined in Equation ([e3.2]), and

$$j(x,t) = \frac{i\,\hbar}{2\,m}\left(\psi\,\frac{\partial\psi^*}{\partial x} - \psi^*\,\frac{\partial\psi}{\partial x}\right)$$

is known as the *probability current*. Note that x is real. Equation ([epc]) is a *probability conservation equation*. According to this equation, the probability of a measurement of x lying in the interval a tob evolves in time due to the difference between the flux of probability into the interval [i.e., j(a,t)], and that out of the interval [i.e.,j(b,t)]. Here, we are interpreting j(x,t) as the flux of probability in the +x-direction at position x and time t.

Note, finally, that not all wavefunctions can be normalized according to the scheme set out in Equation ([e3.4]). For instance, a plane-wave wavefunction

$$\psi(x,t) = \psi_0\,\mathrm{e}^{\,\mathrm{i}\,(k\,x-\omega\,t)}$$

is not square-integrable, and, thus, cannot be normalized. For such wavefunctions, the best we can say is that

$$P_{x\,\in\,a:b}(t) \propto \int_a^b |\psi(x,t)|^{\,2}\,dx.$$

RESULTS AND DISCUSOES:

According to Bahcerlard, uA very geometric vision, a very analytical vision, an aesthetic judgment that involves working terms—there are so many reasons that link participation to elemental cosmic forces.[11].ExposedThe demonstrations of the Theoretical Framework, now in Results and Discussion, will be...Examples shown include those below.The meaning of probability for normalization, throughThe current equation, for example, which is widely applied in Quantum Field Theory, is even more commonly used, being applied in this form in...equations with Mikowski spaces and parameters, to proceed in Quantum Field Theory, such as in the study of Particles and Antiparticles, such as creation particles.theand annihilation, in Symmetry and Supersymmetry, of Quantum Electrodynamics.

Example[9],to Applying the Basel Problem:

Now, oneA practical example is the time-dependent wave function, shown below as an example of a trigonometric equation, as a normalization solution for a confined particle.The demonstration of the resolution is shown below.NNormalization of the Wave Function.

The normalized wavefunction of a particle is

$$\psi(x,t) = A\,e^{-i\omega t}\sin\left(\pi x/L\right)$$

Find the expectation value of position.

Strategy

Substitute the wavefunction into Equation and evaluate. The position operator introduces a multiplicative factor only, so the position operator need not be "sandwiched."

Solution

First multiply, then integrate:

$$\begin{aligned}1 &= \int_0^L dx\, \psi^*(x)\psi(x) \\ &= \int_0^L dx\, \left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) \\ &= A^2\int_0^L dx\, \sin^2\frac{\pi x}{L} \\ &= A^2\frac{L}{2} \\ \Rightarrow A &= \sqrt{\frac{2}{L}}.\end{aligned}$$

The expectation value of position is

$$\begin{aligned}\langle x\rangle &= \int_0^L dx\, \psi^*(x)x\psi(x) \\ &= \int_0^L dx\, \left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)x\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) \\ &= A^2\int_0^L dx\, x\sin^2\frac{\pi x}{L} \\ &= A^2\frac{L^2}{4} \\ \Rightarrow A &= \frac{L}{2}.\end{aligned}$$

The expectation value of momentum in the x-direction also requires an integral. To set this integral up, the associated operator must— by rule—act to the right on the wavefunction :

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}Ae^{-i\omega t}\sin\frac{\pi x}{L}$$
$$= -\frac{\hbar^2}{2m}Ae^{-i\omega t}\frac{d^2}{dx^2}\sin\frac{\pi x}{L}$$
$$= \frac{Ah^2}{8mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}.$$

Therefore, the expectation value of momentum is

$$\langle p\rangle = \int_0^L dx\left(Ae^{+i\omega t}sin\frac{\pi x}{L}\right)\left(-i\frac{Ah}{2L}e^{-i\omega t}cos\,\frac{\pi x}{L}\right)$$
$$= -i\frac{A^2h}{4L}\int_0^L dx\,\sin\,\frac{2\pi x}{L}$$
$$= 0.$$

The function in the integral is a sine function with a wavelength equal to the width of the well, **L**—an odd function about x=L/2 . As a result, the integral vanishes.

The expectation value of kinetic energy in the **x**-direction requires the associated operator to act on the wavefunction:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}Ae^{-i\omega t}\sin\frac{\pi x}{L}$$
$$= -\frac{\hbar^2}{2m}Ae^{-i\omega t}\frac{d^2}{dx^2}\sin\frac{\pi x}{L}$$
$$= \frac{Ah^2}{8mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}.$$

Thus, the expectation value of the kinetic energy is

$$\begin{aligned}\langle K\rangle &= \int_0^L dx\left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(\frac{Ah^2}{8mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}\right)\\ &= \frac{A^2h^2}{8mL^2}\int_0^L dx\,\sin^2\frac{\pi x}{L}\\ &= \frac{A^2h^2}{8mL^2}\frac{L}{2}\\ &= \frac{h^2}{8mL^2}.\end{aligned}$$

The average position of a large number of particles in this state is L/2. The average momentum of these particles is zero because a given particle is equally likely to be moving right or left. However, the particle is not at rest because its average kinetic energy is not zero. Finally, the probability density is

$$|\psi|^2 = (2/L)\,\sin^2(\pi x/L).$$

This probability density is largest at location L/2 and is zero at x=0 and at x=L . Note that these conclusions do not depend explicitly on time.

Repeating the Example, below[8]

The example below could beconsidered a more explicit form in the epistemology ofanalysisConceptual, such as the Gaussian type of wave function. It generalizes this type of function to all types, in order to apply an analogous solution, as was Euler's solution to the Basel Problem.

For example, suppose that we wish to normalize the wavefunction of a Gaussian wave-packet, centered on x=x0 , and of characteristic width sigma (see Section [s2.9]): that is,

$$\psi(x) = \psi_0\,\mathrm{e}^{-(x-x_0)^2/(4\,\sigma^2)}.$$

In order to determine the normalization constant , we simply substitute: ψ_0,

Equation ([e3.5]) into Equation ([e3.4]) to obtain

$$|\psi_0|^2 \int_{-\infty}^{\infty} e^{-(x-x_0)^2/(2\sigma^2)}\, dx = 1.$$

Changing the variable of integration to

$$y = (x - x_0)/(\sqrt{2}\,\sigma)$$

we get

$$|\psi_0|^2 \sqrt{2}\,\sigma \int_{-\infty}^{\infty} e^{-y^2}\, dy = 1.$$

However,

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi},$$

which implies that

$$|\psi_0|^2 = \frac{1}{(2\pi\,\sigma^2)^{1/2}}.$$

Hence, a general normalized Gaussian wavefunction takes the form

$$\psi(x) = \frac{e^{i\,\varphi}}{(2\pi\ \sigma^2)^{1/4}}\, e^{-(x-x_0)^2/(4\,\sigma^2)},$$

where

$$\varphi\ \mathrm{i}$$

is an arbitrary real phase-angle.

It is important to demonstrate that if a wavefunction is initially normalized then it stays normalized as it evolves in time according to Schrödinger's equation. If this is not the case then the probability interpretation of the wavefunction is untenable, because it does not make sense for the probability that a measurement of yields any possible outcome (which is, manifestly, unity) to change in time. Hence, we require that

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\psi(x,t)|^{2}\,dx=0,$$

The intention is simply to apply.rthe problem of Basel, like all its typesof solutions, in addition to Euler's solution, in wave functions of the Schrödinger equation forpossible and future analyses, whether for an epistemological approachor didactics, in solving exercises, with the methods of solutionfrom Euler to the Basel problem, but also asthe geometric construction that Euler carried out, the approach, starting from methods ofPythagorean school, such as trigonometric relations and equalities.and inequalities.ANDThe study of phenomena, that is, of what appears to consciousness, of what is given. It involves exploring this given fact, the very thing that is perceived, thought about, and spoken of, avoiding the formulation of hypotheses, both about the link that unites the phenomenon with the being of which it is a phenomenon, and about the link that unites it with the Self for whom it is a phenomenon.[10].

Finally, it should be noted that Concepts of integration by parts, with their trigonometric transformations, as in the example below, only.emphasis.

Evaluate

$$\int e^{x}\ \sin\ x\ dx.$$

$u = e^{x}$	$du = e^{x}\ dx$
$dv = \sin x\ dx$	$v = -\cos x$

$$\int e^{x}\ \sin\ x\, dx = -e^{x}\ \cos\ x + \int e^{x}\ \cos, x\, dx$$

We have

$$\int e^{x}\ \sin\ x\, dx = -e^{x}\ \cos\ x + \int e^{x}\ \cos, x\, dx$$

Now try integration by parts again:

$u = e^x$	$du = e^x\, dx$
$dv = \cos x\, dx$	$v = \sin x$

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx.$$

It seems as if we are back to where we started, however with a clever move, the answer appears.

Let

$$I = \int e^x \sin x\, dx.$$

$$I = -e^x \cos x + e^x \sin x - I.$$

$$2I = -e^x \cos x + e^x \sin x.$$

$$I = \frac{-e^x \cos x + e^x \sin x}{2} + C.$$

$$\int e^x \sin x\, dx = \frac{-e^x \cos x + e^x \sin x}{2} + C.$$

CONCLUSION

Solving exercises, using Wave Function Analysis as an example, applying the solution of...Euler and the Basel ProblemThe meaning of the Wave Function is found in the Epistemology of Teaching, according to the Equation ofSchrödinger in order to solve the exercises of Quantum Mechanics conceptually. Likewise, the passagesnsExamples from differential and integral calculus serve as examples for the transformations and steps necessary for...This work analyzes the meaning of the Wave Function. In this paper, only a theoretical review approach was undertaken, due to the fact that it is an article of less than 20 pages. The theory of the Wave Function and the Basel Problem were presented in a way that was clear and concise.Sure, the type of application that should be used...perform, for a later analysis of the Wave Function in the future of the realizationRegarding the exercise of the Schrödinger Equation, what is it or what does it mean?With this, it becomes more feasible toresolutionsexercises that require, for example, a Technical-Scientific assignment using the Schrödinger Equation, because each proper wave equation will know the meaning of a given Wave Function applied to the Schrödinger Equation, as in Quantum Well Analysis, for example, theSemiconductors or Technical-Scientific Tunneling Works in Quantum Mechanics, or in Superconductors. Therefore, the present work is only a point ofThis is a starting point for approaching the meaning of the Wave Function as applied to the Wave Equation, which requires the following work:In the context of mathematical tools, such as Euler's approach in solving the Basel Problem with trigonometric constructions, as well as many solutions to mathematical theories, the starting point is...tools of the School of Athens and ofThe Pythagorean school, in the early stages of Geometric Analysis, as well as an approach toResolutions of Roots, as was the case in the context of the Basel School of Problems in European Mathematics during the Renaissance and the Modern World. The next article will contain a demonstration of the meaning of wave as an example, based on the Basel problem.

Proving the central problem of this article.

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