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ON SOLVING NON-HOMOGENEOUS TERNARY HIGHER DEGREE DIOPHANTINE EQUATION

$$x^2 + y^2 = (a^2 + b^2)\, z^{2s+1}$$

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ABSTRACT

The non-homogeneous ternary higher degree Diophantine equation given by $x^2 + y^2 = (a^2 + b^2)\, z^{2s+1}$ is analyzed for its patterns of non-zero distinct integral solutions.

KEYWORDS: Ternary higher degree equation, Non- Homogeneous equation, Integral Solutions.

INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. For the sake of clear understanding by the readers, one may refer the varieties of Cubic, Quintic ,Heptic and Nonic Diophantine equations with multi variables [1-33]. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree (2s+1) with three unknowns given by $x^2 + y^2 = (a^2 + b^2)\ z^{2s+1}$.

Method of analysis

The non-homogeneous ternary higher degree Diophantine equation to be solved for its distinct integer solutions is

$$x^2 + y^2 = (a^2 + b^2)\ z^{2s+1} \qquad (1)$$

The process of obtaining patterns of integer solutions to (1) is analyzed below:

Pattern 1

Choosing

$$x = z^s X, y = z^s Y \quad (2) \text{ in}$$

(1) leads to the non-homogeneous ternary quadratic equation

$$X^2 + Y^2 = (a^2 + b^2) z \quad (3)$$

Assume

$$z = (p^2 + q^2)^n \quad (4)$$

Substitute (4) in (3).Utilizing factorization and equating positive factors, we have

$$\begin{aligned} X + i\,Y &= (a + i\,b)(p + i\,q)^n \\ &= (a + i\,b)[f(p,q) + i\,g(p,q)] \end{aligned} \quad (5) \text{ where}$$

$$\begin{aligned} f(p,q) &= \frac{1}{2}[(p + i\,q)^n + (p - i\,q)^n] \\ g(p,q) &= \frac{1}{2i}[(p + i\,q)^n - (p - i\,q)^n] \end{aligned} \quad (6)$$

Equating the real and imaginary parts in (5), we get

$$\begin{aligned} X &= a\ f(p,q) - b\ g(p,q) \\ Y &= b\,f(p,q) + a\ g(p,q) \end{aligned}$$

In view of (2) , the integer solutions to (1) are given by

$$\begin{aligned} x &= (p^2 + q^2)^{sn}[a\,f(p,q) - b\,g(p,q)] \\ y &= (p^2 + q^2)^{sn}[b\,f(p,q) + a\,g(p,q)] \end{aligned}$$

(5) together with (4). It is worth to mention that the values of a, b, p, q are chosen such that

$a\,f(p,q) \neq b\,g(p,q)$.

Special cases

(1) The choice n=1 gives the integer solutions to (1) as

$$x = (p^2 + q^2)^s (a\,p - b\,q)$$
$$y = (p^2 + q^2)^s (b\,p + a\,q)$$
$$z = (p^2 + q^2)$$

A few numerical solutions are given in the Table-1 below:

Table-1-Numerical solutions.

a	b	p	q	x	y	z
1	2	2	3	$-4*13^s$	$7*13^s$	13
2	4	1	3	$-10*10^s$	$10*10^s$	10
5	3	4	2	$14*20^s$	$22*20^s$	20
4	3	3	2	$6*13^s$	$17*13^s$	13

From the above Table-1,it is seen that

$$q\,(a\,x + b\,y) = p\,(a\,y - b\,x) \quad (2)$$

The choice n=2 gives the integer solutions to (1) as

$$x = (p^2 + q^2)^{2s} [a\,(p^2 - q^2) - 2\,b\,p\,q]$$
$$y = (p^2 + q^2)^{2s} [b\,(p^2 - q^2) + 2\,a\,p\,q]$$
$$z = (p^2 + q^2)^2$$

A few numerical solutions are presented in Table-2 below:

Table-2-Numerical solutions.

a	b	p	q	x	y	z
1	2	2	3	$-29*13^{2s}$	$2*13^{2s}$	13^2
2	4	1	3	$-40*10^{2s}$	$-20*10^{2s}$	10^2
5	3	4	2	$12*20^{2s}$	$116*20^{2s}$	20^2
4	3	3	2	$-16*13^{2s}$	$63*13^{2s}$	13^2

It is observed from the above Table-2 that

$$(a\,x + b\,y)^2 + (a\,y - b\,x)^2 = [(a^2 + b^2)(p^2 + q^2)\,z^s]^2$$

Pattern 2

Taking

$$y = bkz^s \quad (7)$$

in (1) ,it gives

$$x^2 = z^{2s}[(a^2+b^2)z - b^2k^2] \quad (8)$$

After some algebra, it is seen that (8) is satisfied by

$$\begin{aligned} z &= z_n = (a^2+b^2)n^2 + 2akn + k^2 \\ x &= x_n = [(a^2+b^2)n^2 + 2akn + k^2]^s[(a^2+b^2)n + ak] \end{aligned} \quad (9)$$

From (7), we get

$$y = y_n = [(a^2+b^2)n^2 + 2akn + k^2]^s \quad (10)$$

Thus,(9) & (10) satisfy (1).

CONCLUSION

In this paper, we have made an attempt to obtain patterns of integer solutions to ternary higher degree Diophantine equation $x^2 + y^2 = (a^2+b^2)z^{2s+1}$. To conclude, one may search for integer solutions to other choices of homogeneous or non-homogeneous higher degree Diophantine equations with multiple variables.

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